Signal Processing for MRI

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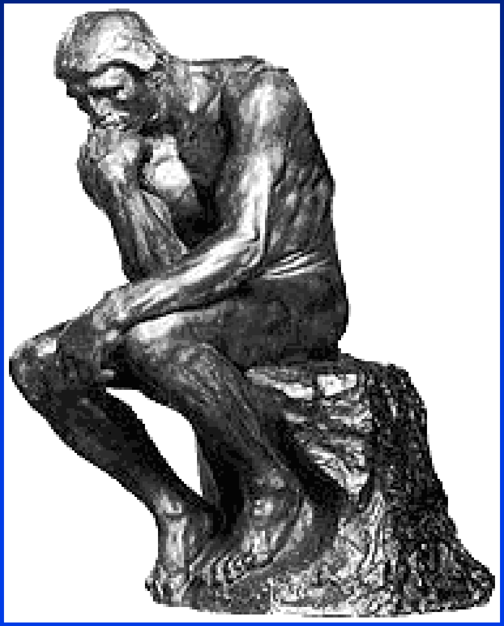
A Philosophical Debate

Do we live in a digital world or an analog world?



Auguste Rodin, bronze, ca. 1880

An Engineering Reality

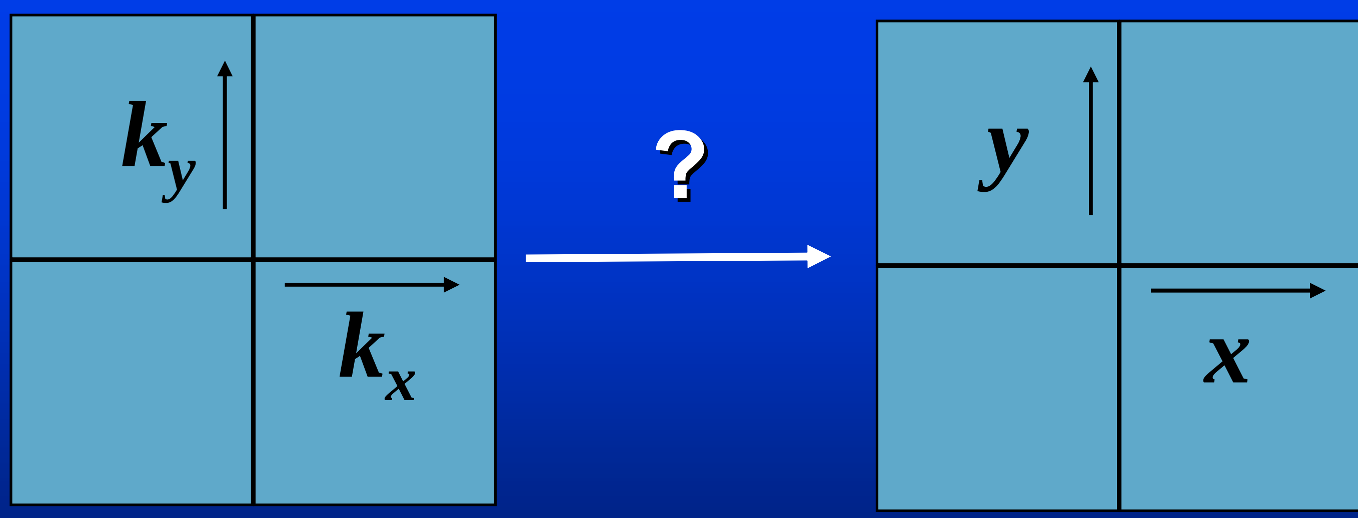
We live in a digitized world.

Andrew Lipson, LEGO bricks, ca. 2000

Fundamental fact:

- *MRI data is acquired in k-space*
- x-space is just a derived quantity
  (which we happen to be interested in)

Therefore, we need to understand:



Data

Image

Plan: to demonstrate that

• The basic concepts of time/frequency signal processing can be carried over to MRI

• Δk_x and Δk_y are the relevant sampling intervals

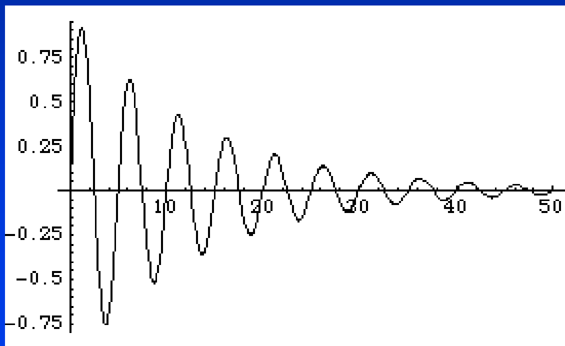
• The imaging equation defines the transformation between conjugate variables--Fourier

• Sampling and other operations on data are performed in k-space; the convolution theorem supplies the resulting effect in the image

• Both DSP and physical effects must be considered

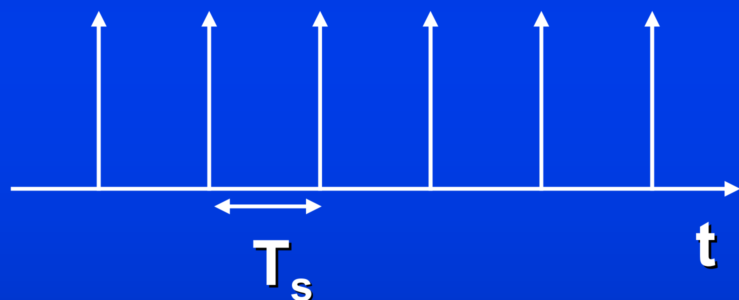
Digitization of a Time-Domain Analog Signal

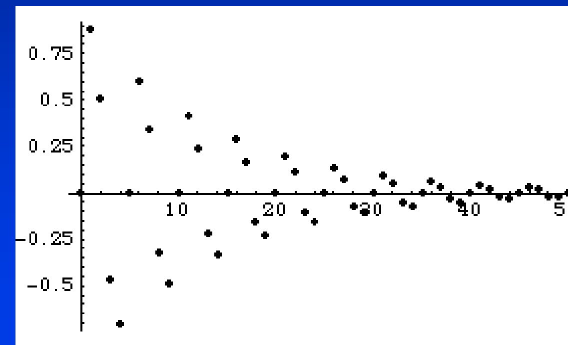
Sampling Interval T_s



$g(t)$

$$\bullet \quad \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) \quad =$$



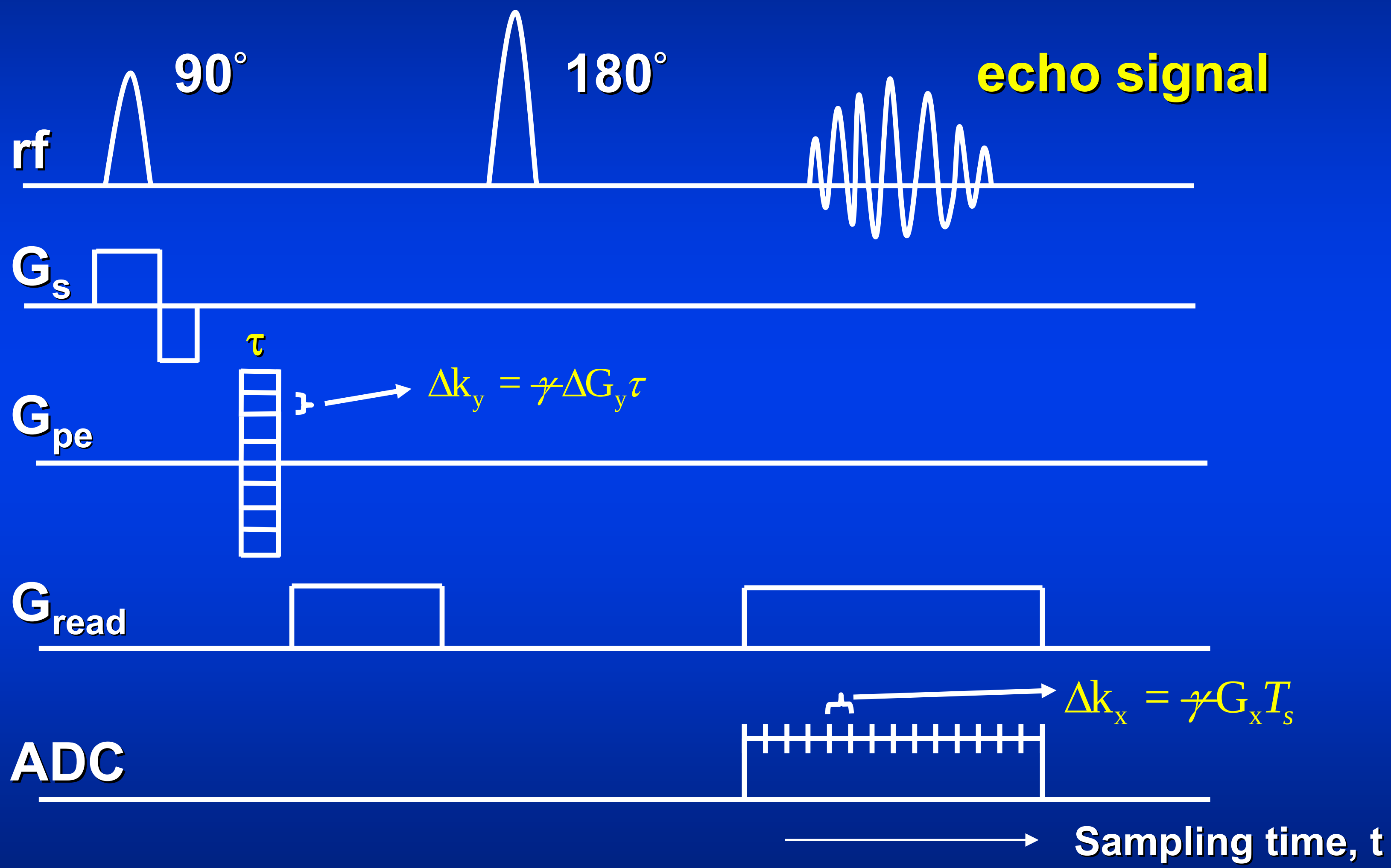


$g_{samp}(t)$

$$g_{samp}(t) = g(t) \bullet \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{+\infty} g(nT_s)\delta(t - nT_s)$$

Data spaced at intervals T_s

Sampling during MRI signal acquisition

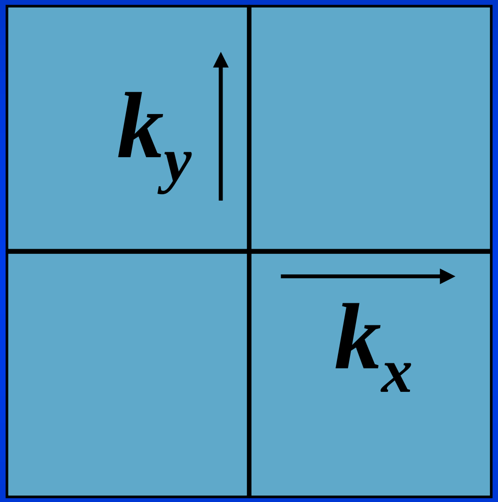


Sampling in k-space

Read direction

$$k_x = \bar{\gamma} G_x t$$

$$\Delta k_x = \bar{\gamma} G_x T_s$$



Phase direction

$$k_y = \bar{\gamma} G_y \tau$$

$$\Delta k_y = \bar{\gamma} \Delta G_y \tau$$

For both dimensions: data is spaced at intervals Δk

$$\bar{\gamma} \equiv \gamma / 2\pi$$

Data in k-space is (usually) regularly sampled on a grid.

This sampling is entirely analogous to sampling of time-domain data:
intervals are **Δk_x and Δk_y** ***instead of interval*** **T_s**

Significance of this:

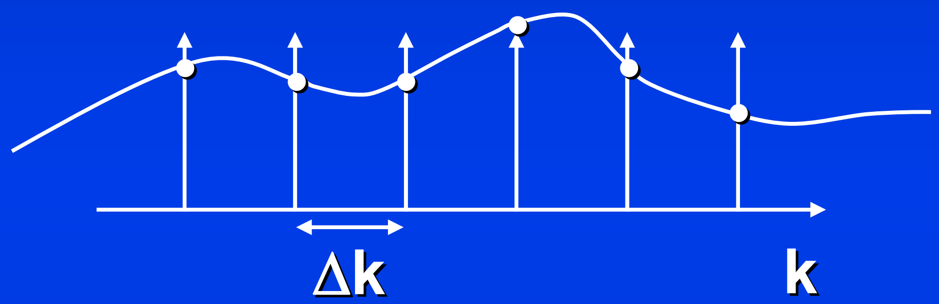
- **From a post-processing point of view, read and phase directions in MRI can be handled in an identical fashion**

- **Much of what you already know about signal processing of sampled time-domain signals can be immediately carried over to MRI**

The k-space sampling function is written:

$$Comb(k;\Delta k) = \sum_{n=-N/2}^{+N/2} \delta(k - n\Delta k)$$

N, number of sampled points in k_x or k_y



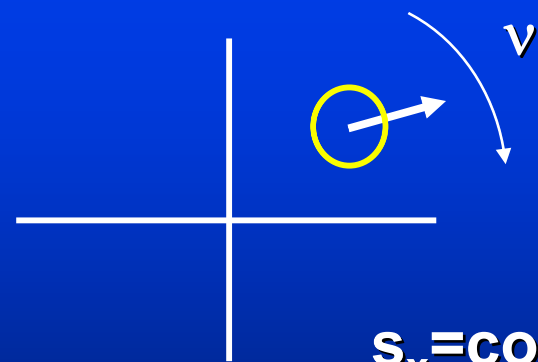
k-space data are numbers assigned to each grid point:
These are the samples $s_{samp}(k_x, k_y)$

Conceptually, we can consider $s_{samp}(k_x, k_y)$
to be a sampled version of some continuous function, $s(k_x, k_y)$

The above dealt with signal acquisition

To proceed: *consider the physics*

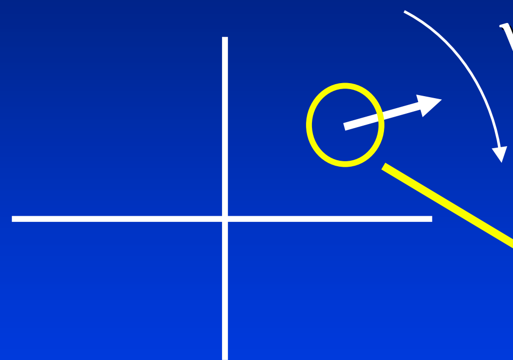
Relationship Between Signal and Precessing Spins During Read



$s_x = \cos(2\pi\nu t)$

$s_y = \sin(2\pi\nu t)$

Signal and Spins During Read



Signal from dx dy: $s(t; x, y)\, dx\, dy = \overbrace{\rho(x, y)}\ \overbrace{e^{-i\,2\pi\,\nu\,t}}\, dx\, dy$

$$\left.\begin{array}{l}\boxed{\nu = \bar{\gamma}\, x\, G_r} \\ \boxed{k_x = \bar{\gamma}\, t\, G_r}\end{array}\right\} \rightarrow \boxed{\nu\, t = k_x\, x}$$

$\iint$ Integrate over all excited spins

$$s(t \equiv k_x x / \nu) = \iint \rho(x, y)\, e^{-i 2\pi k_x x}\, dx dy$$

Consideration of the phase encode gradient leads to the celebrated *imaging equation*

…relates k-space data, $s(k_x, k_y)$ to the image, $\rho(x,y)$

$$s(k_{x,}k_y) = \iint \rho(x,y)\, e^{-i\pi(k_x x + k_y y)}\, dxdy \equiv \Im\left[\rho(x,y)\right]$$

$$\rho(x,y) = \iint s(k_{x,}k_y)\, e^{+i\pi(k_x x + k_y y)}\, dk_x dk_y \equiv \Im^{-1}\left[s(k_{x,}k_y)\right]$$

Note: the Fourier transform arises from the physics

Combine Fourier transforms with convolution to make use of the all-powerful *Convolution Theorem*

Convolution of x(t) and h(t)

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

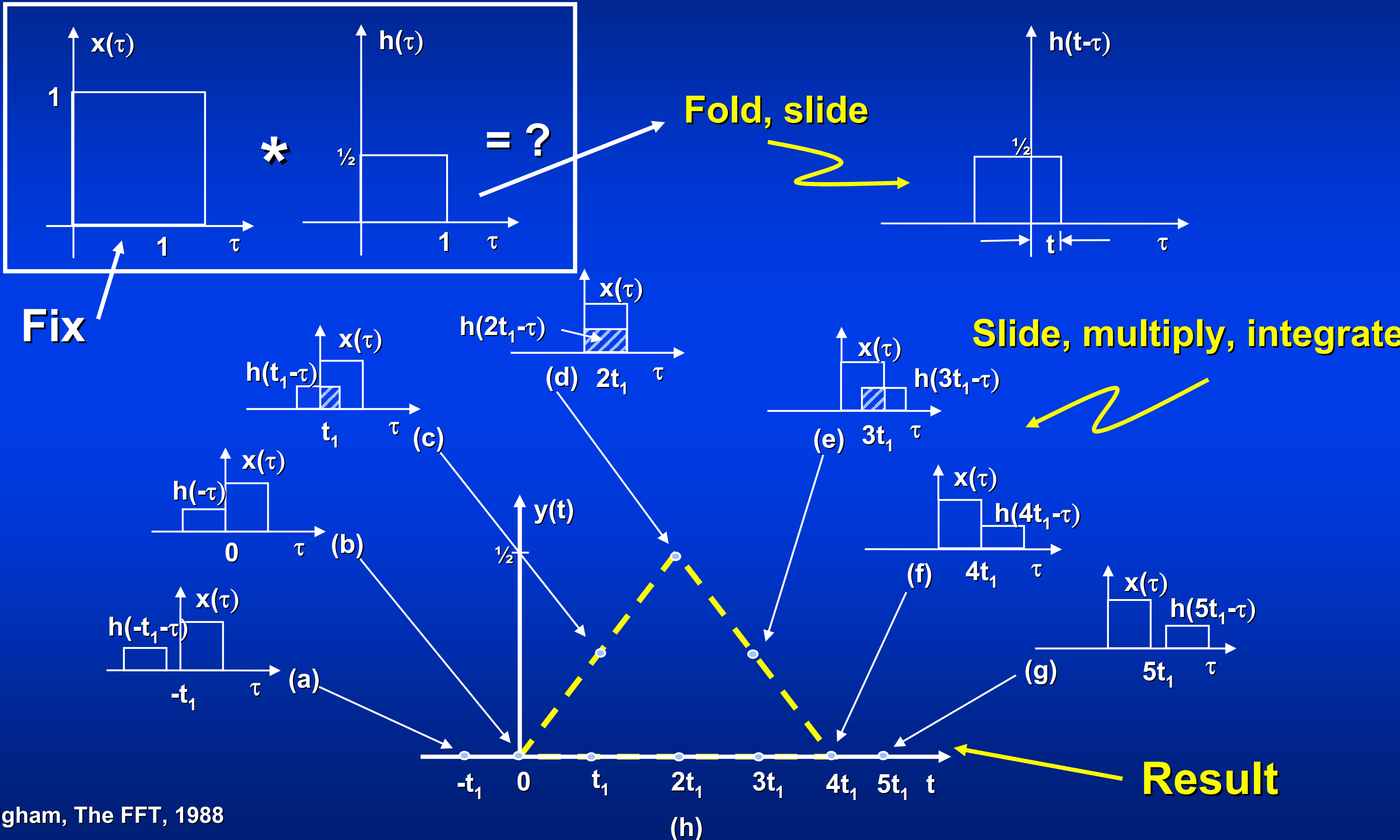
fold → slide → multiply → integrate

Arises naturally when considering:

- the observable effects of intended or unintended actions on data
- digital filters

Convolution of x(t) and h(t) =

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$



Brigham, The FFT, 1988

The Convolution Theorem

Ingredients:

h(t) and g(t), and their Fourier transforms H(ν), G(ν)

$\Im$ = Fourier transform

$\Im^{-1}$ = Inverse Fourier transform

• = multiplication

* = the convolution operator

4 ways of writing the convolution theorem:

I. $\Im\{f*g\} = F \bullet G$

II. $\Im\{f \bullet g\} = F * G$

III. $\Im^{-1}\{F \bullet G\} = f * g$

IV. $\Im^{-1}\{F * G\} = f \bullet g$

Our application is based on the imaging equation:

$$\Im\{\rho(x, y)\} = s(k_x, k_y)$$

$$\Im^{-1}\{s(k_x, k_y)\} = \rho(x, y)$$

Visible effect on the image

Version III. $\Im^{-1}\{s \bullet H\} = \rho \overbrace{* \; h}$

Various non-idealities or filters

Ideal data in k space

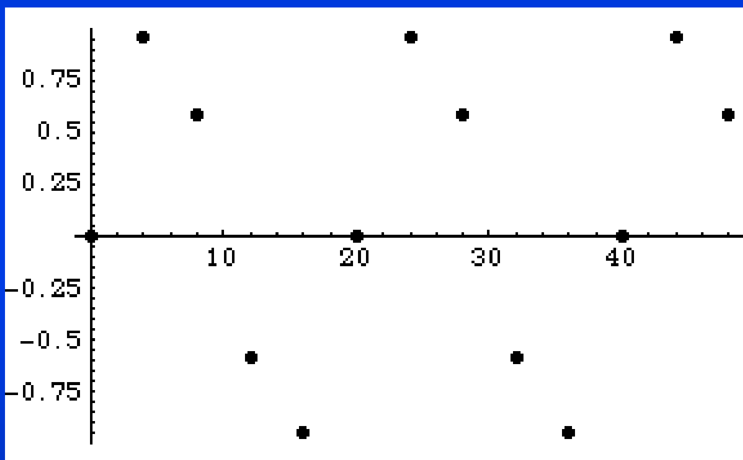
ρ = the ideal image

With this, we can understand the effects that sampling, truncation, and relaxation in k-space have on the image

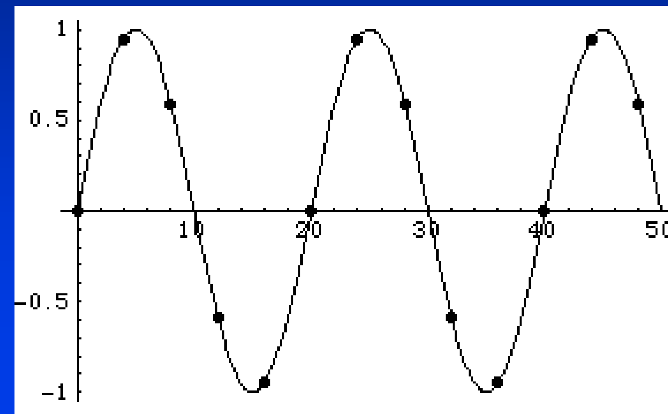
- **Aliasing**
  - *direct sampling effect*
- **The point spread function**
  - *truncation--signal processing*
  - *relaxation--physics*

Aliasing

aka wrap-around, aka fold-over

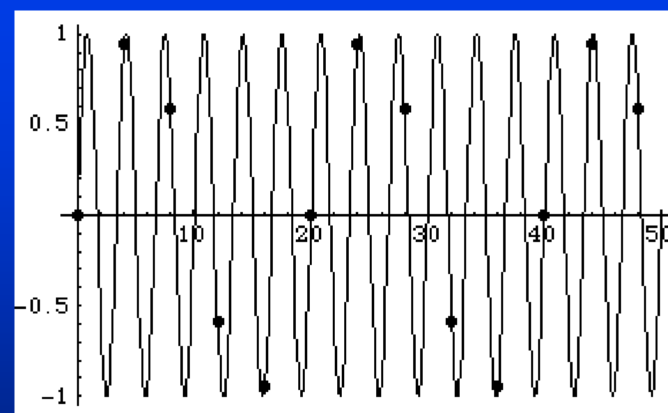


?



Equally good digital choices!

?



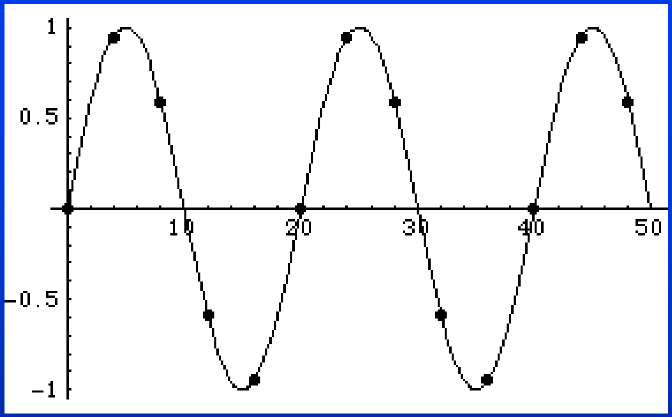
Thus, high frequency sinusoids, when sampled, can be mis-assigned to a lower frequency!

To avoid this, sample at a rate $\nu_S = 1/T_s$ which satisfies

$$\nu_S > 2 \cdot \nu$$

where ν is the frequency of the sinusoid

$$\overleftrightarrow{\quad 1/\nu \quad}$$



This rate, $2 \cdot \nu$ is called the *Nyquist* rate, ν_N

To avoid aliasing: $\nu_S > \nu_N \equiv 2 \cdot \nu$

Fourier decomposition permits extension of this theorem to a general bandlimited ($-\nu_{max}$, ν_{max}) signal, described as:

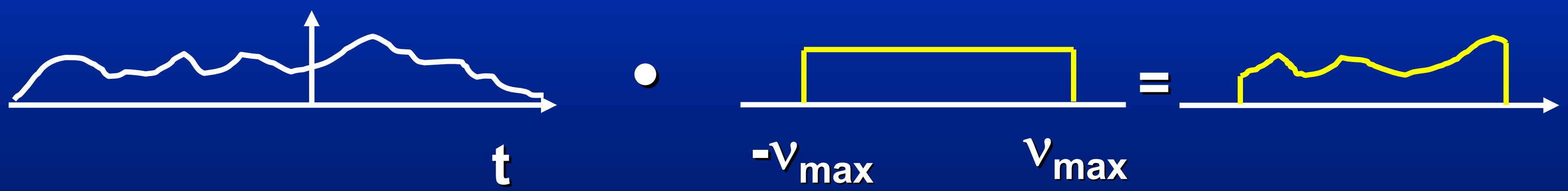
$$g(t) = \int_{-\nu_{max}}^{\nu_{max}} G(\nu) e^{i2\pi\nu t} d\nu$$

Then aliasing is avoided by ensuring

$$\nu_S > \nu_N \equiv 2 \bullet \nu_{max}$$

Note: for a non-bandlimited signal, apply an anti-aliasing prefilter:

Non-bandlimited signal prefilter bandlimited signal

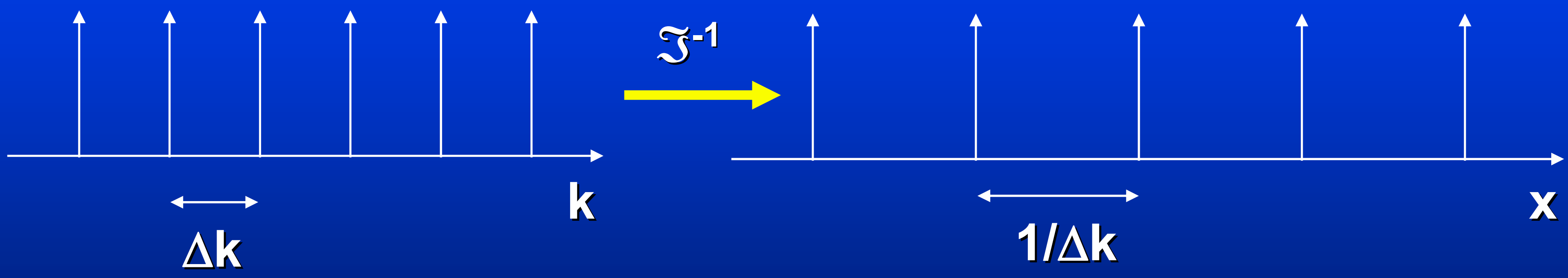


The convolution theorem defines the effect on the image of sampling

$$Comb(k;\Delta k) = \sum_{-\infty}^{\infty} \delta(k - m\Delta k)$$

A straightforward calculation shows:

$\mathfrak{I}^{-1}\{Comb(k;\Delta k)\} = 1/\Delta k \bullet Comb(x; 1/\Delta k)$



We can now calculate:

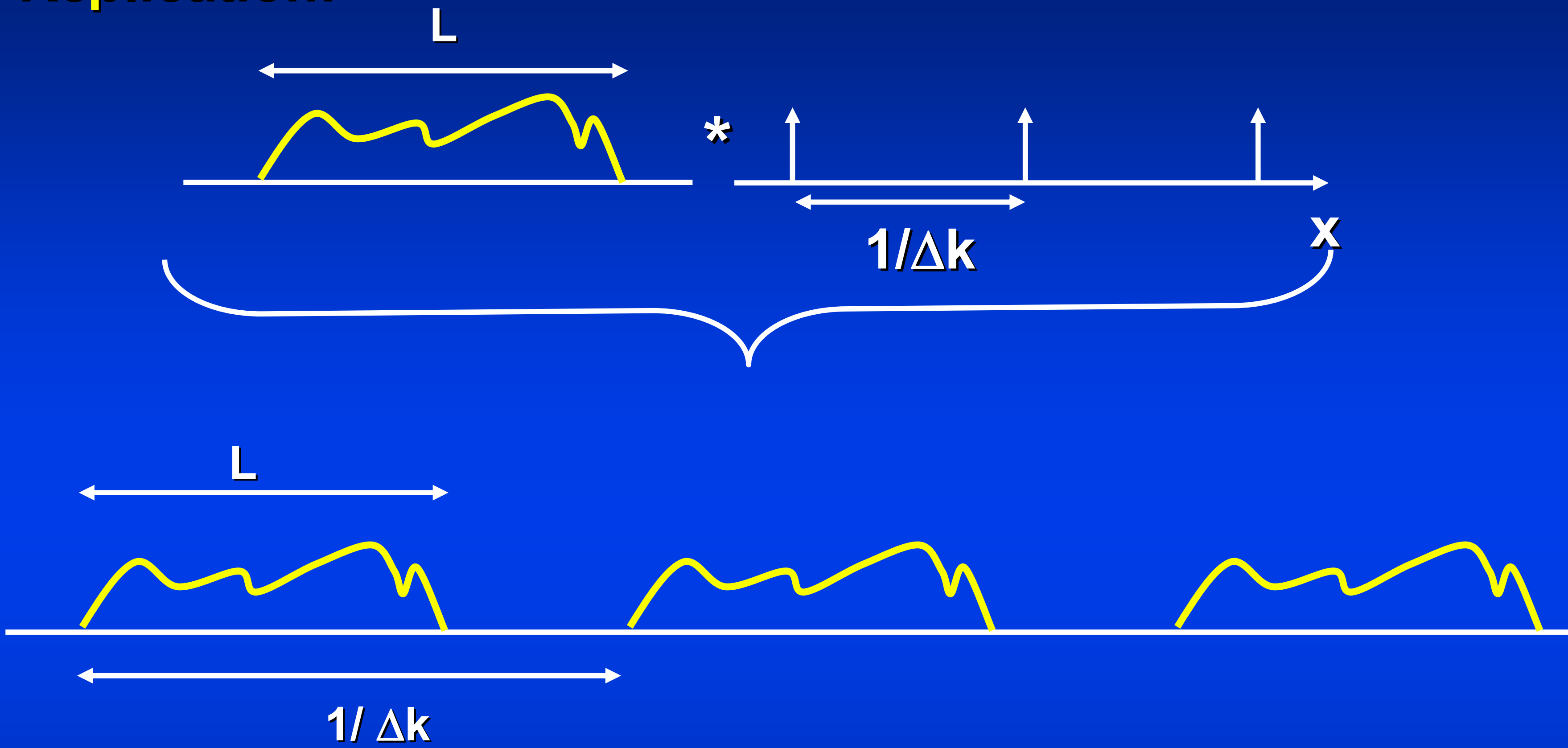
$\rho_{samp}(x) = \Im^{-1}\{s(k) \bullet Comb(k; \Delta k)\}$

$= \Im^{-1}\{s(k)\} * \Im^{-1}\{Comb(k; \Delta k)\}$

$= \quad \rho(x) * Comb(x; 1/\Delta k)$

Obtain replicates, spaced at a distance 1/ Δk apart

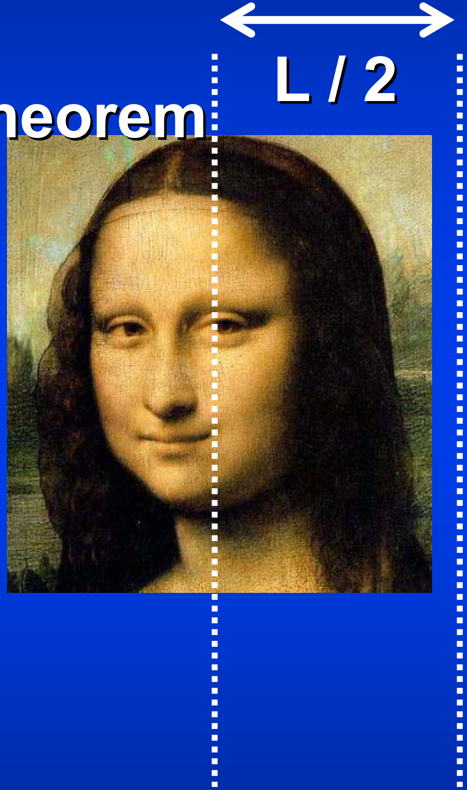
Replication:



Provided 1/ Δk > L, there is no overlap and correct reconstruction is possible

Using only the convolution theorem, we found that we can avoid aliasing by selecting

- $\Delta k_x < 1/L_x$
- $\Delta k_y < 1/L_y$



This is *equivalent to* the Nyquist sampling theorem

i) $\Delta k_x = \cancel{\gamma} G_x T_s$ (definition)

ii) $\Delta k < 1/L$ (the condition derived above)

i) and ii) $\Rightarrow$

iii) $\cancel{\gamma} G_x T_s < 1/L$

which can be written:

iv) $T_s < \dfrac{1}{\cancel{\gamma} G_x L}$

$\nu = 0$ $\quad \nu_{max} = \cancel{\gamma} \dfrac{L}{2} G_x$

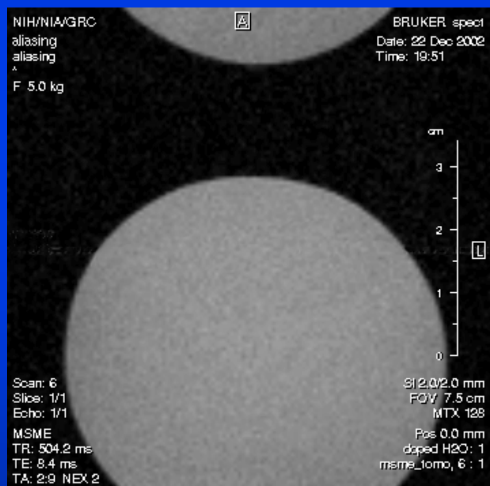
using the value of ν_{max}, we obtain

v) $T_s < 1/(2\,\nu_{max})$ which can also be written: $\nu_s > 2\,\nu_{max}$ = ***The Nyquist condition***

Thus, to fit the entire object into the image, one needs to sample in k-space such that $\Delta k < 1 / L$ is satisfied

Δk is called the FOV

This was derived for the read direction, but identical considerations apply in the phase encode direction

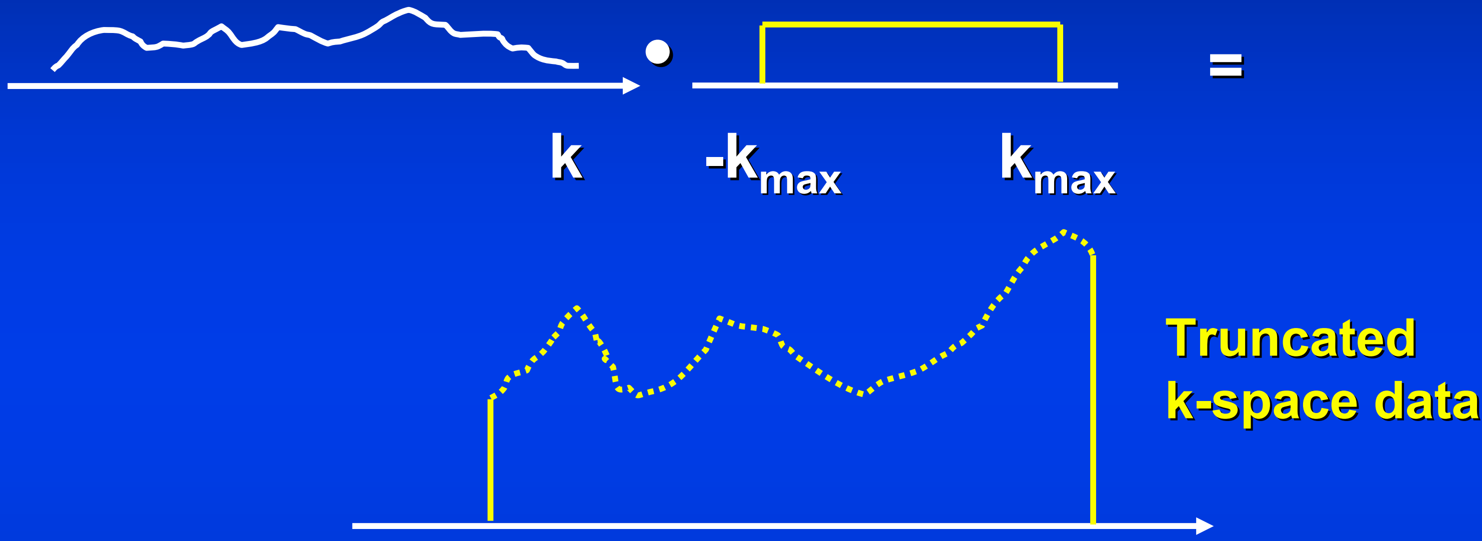


FOV = 7.5 cm
Aliased in phase encode

FOV = 15 cm
Non-aliased

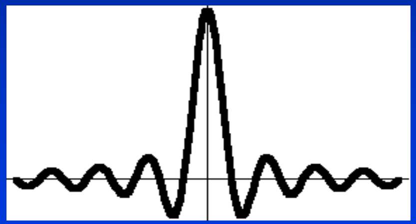
Point Spread Function Due to *Signal Processing*

Actual data are samples from truncated k-space



Actual: $s(k) \rightarrow \sqcap \cdot s(k) = s_{trunc}(k)$

The convolution theorem can help define the result of this truncation

We will use: $\Im^{-1}\{$  $(k)\} =$ 

Sin(x)/x
also known as
Sinc(x)

The resulting 1-D image is given by:

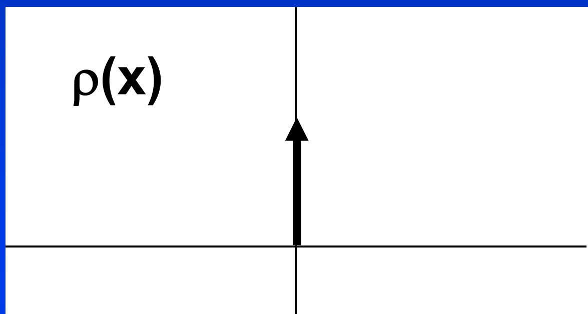
$$\rho_{trunc}(x) = \Im^{-1}\{s_{trunc}(k)\}$$

$$= \Im^{-1}\{\,Rect(k) \cdot s(k_x)\}$$

$$= \Im^{-1}\{\,Rect(k)\} * \Im^{-1}\{s(k)\}$$

$$= \frac{\sin(2\pi k_{\max} x)}{2\pi k_{\max} x} * \rho(x)$$

Therefore, a delta function density distribution in one dimension becomes:



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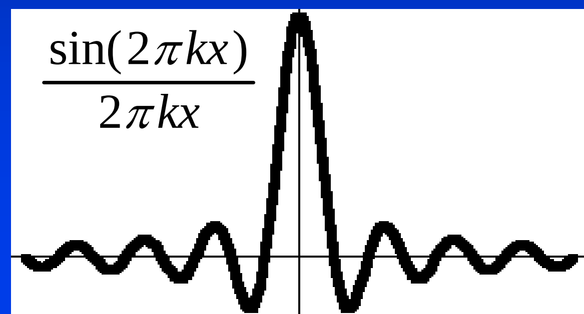

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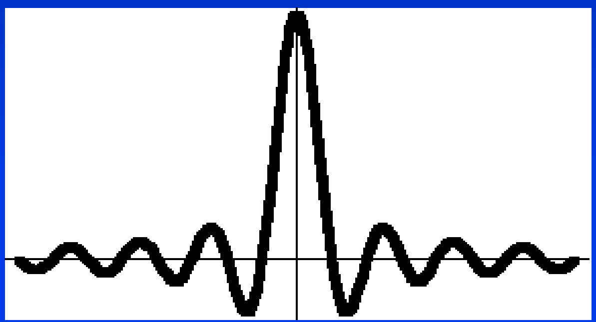
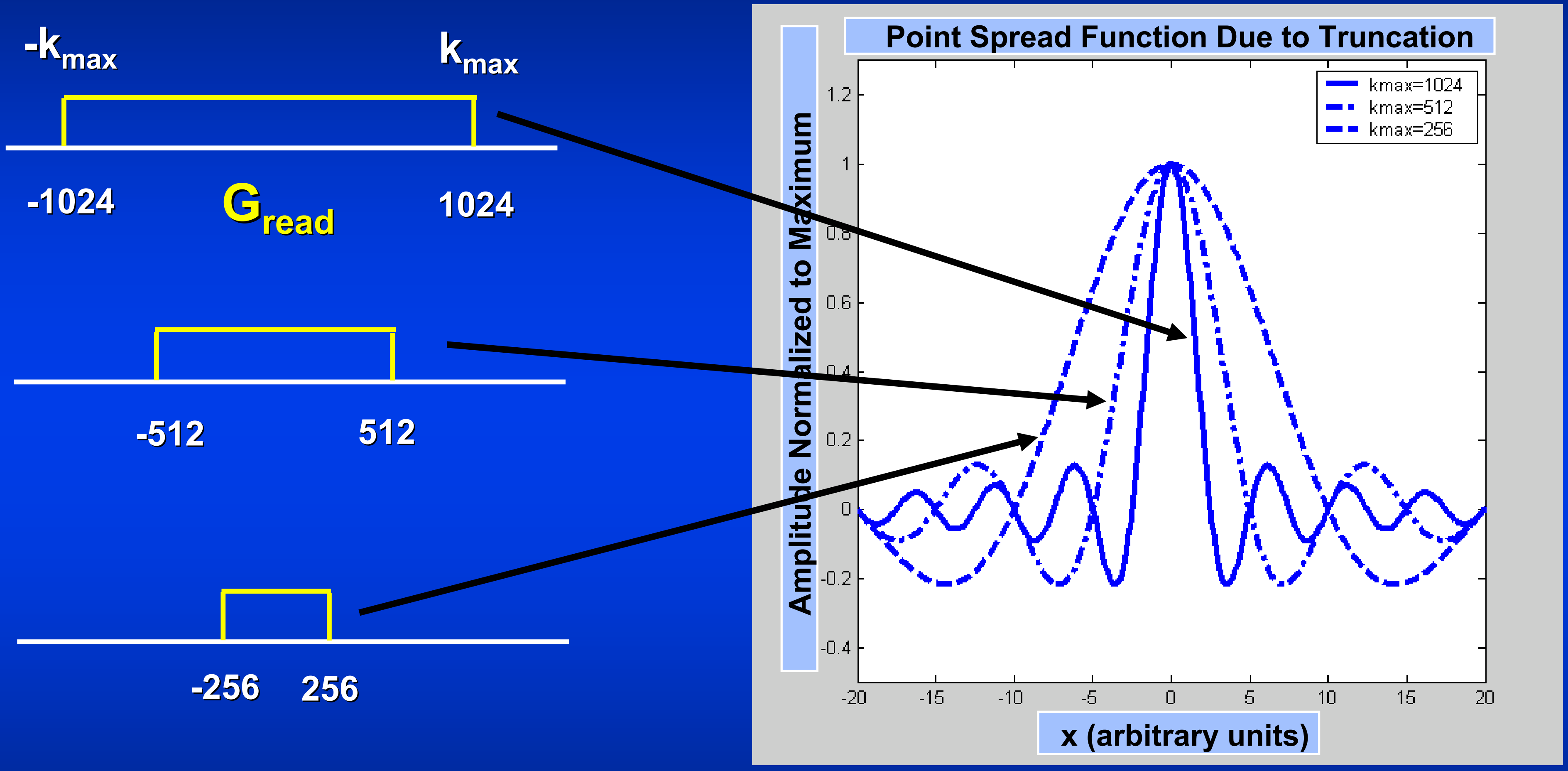


Ideal point object

Smearing from truncation

Actual image:
blurred

More truncation gives more blurring



We can now calculate the combined effects of sampling and truncation:

$\rho_{samp,\ trunc}(x) = \Im^{-1}\{s(k) \bullet Rect(k) \bullet Comb(k; \Delta k)\}$

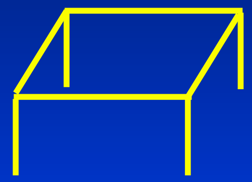
$= \Im^{-1}\{s(k)\} * \Im^{-1}\{Rect(k)\} * \Im^{-1}\{Comb(k; \Delta k)\}$

$$= \rho(x) * Comb(x; 1/\Delta k) * \frac{1}{\Delta k} 2k_{\max} \frac{\sin(2\pi k_{\max} x)}{2\pi k_{\max} x}$$

Obtain:

- **replication, spaced at a distance $1/\Delta k$ apart**
- **smearing**

In two dimensions: two dimensional truncation!

$s(k_x, k_y) \rightarrow$  $\bullet\ s(k_x, k_y) = s_{trunc}(k_x, k_y)$

The image is given by:

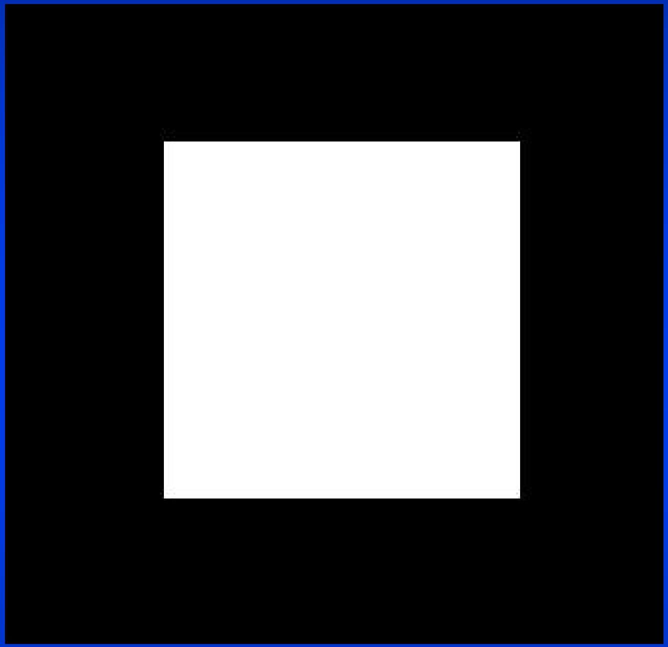
$$\rho_{trunc}(x,y) = \Im^{-1}\left\{s_{trunc}(k_x,k_y)\right\}$$

Conv Thm $\rightarrow$

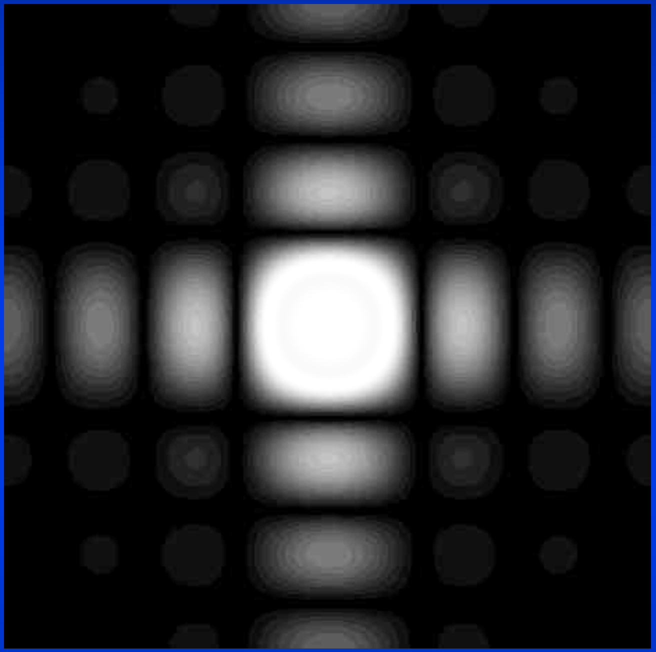
$$= \Im^{-1}\left\{rect(k_x,k_y)\right\} * \Im^{-1}s(k_x,k_y)$$

$$= \frac{\sin(2\pi k_{x,\max} x)}{2\pi k_{x,\max} x} \bullet \frac{\sin(2\pi k_{y,\max} y)}{2\pi k_{y,\max} y} * \rho(x,y)$$

What does this point spread function look like?



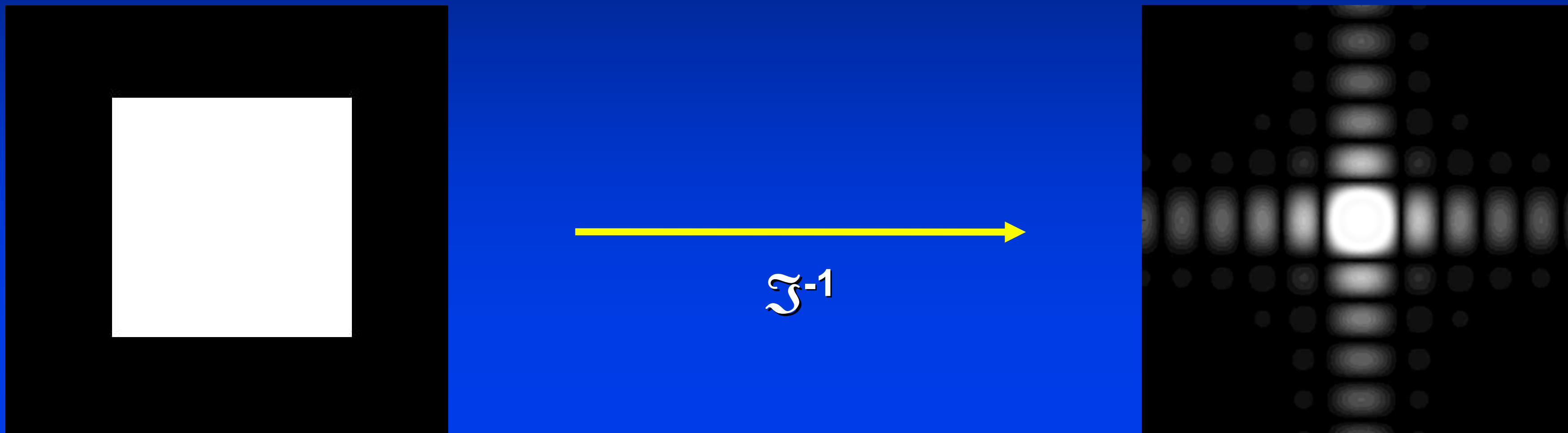
$\xrightarrow{\mathfrak{F}^{-1}}$



Truncation pattern in 2D k-space

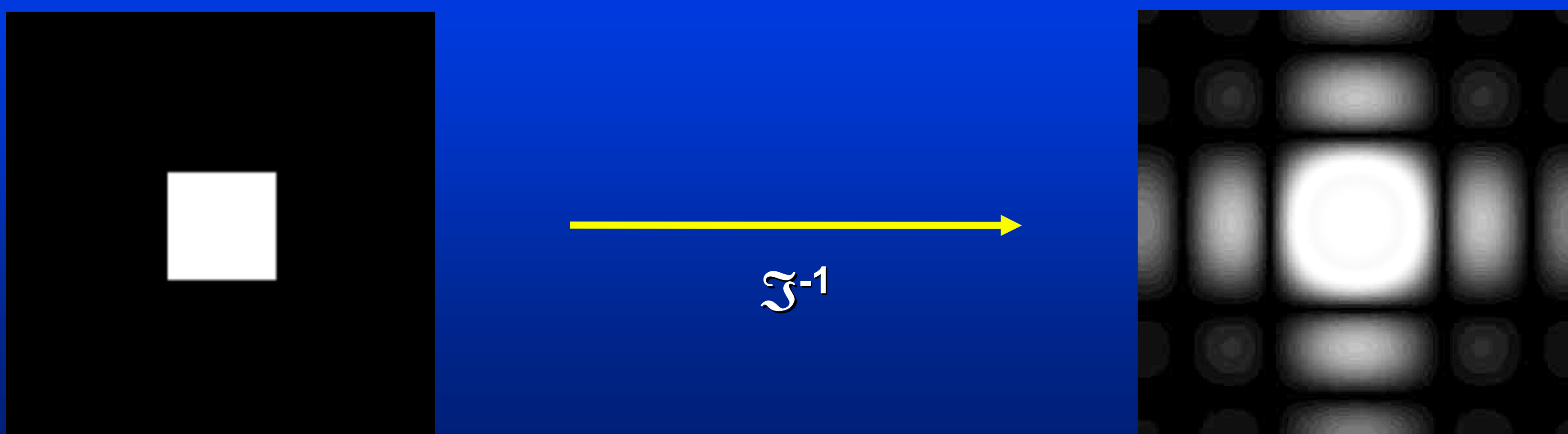
PSF in 2D x-space

As in 1 dimension, width of point spread function is inverse to width of truncation



Truncation pattern in 2D k-space

PSF in 2D x-space



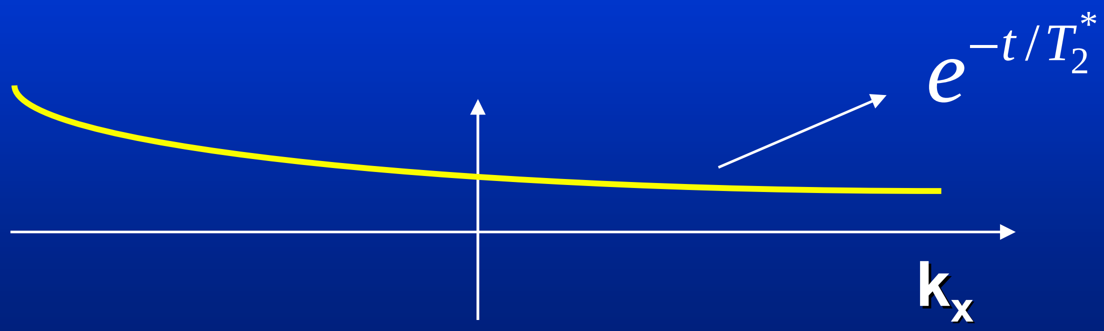
Next example: Point Spread Function Due to *Physics*

The effect of T_2 decay*

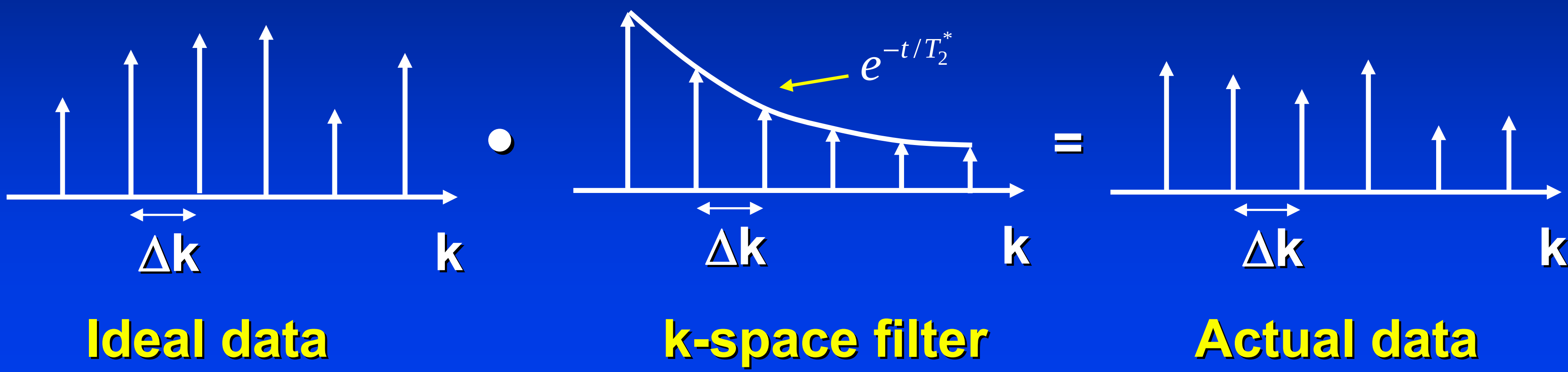
$$1/T_2^{*} = 1/T_2 + 1/T_2'$$

Net decay — Thermodynamic decay — Reversible dephasing

In the gradient echo experiment, both T_2 and T_2' decay start from the beginning of each k-space line, at $-k_{max}$



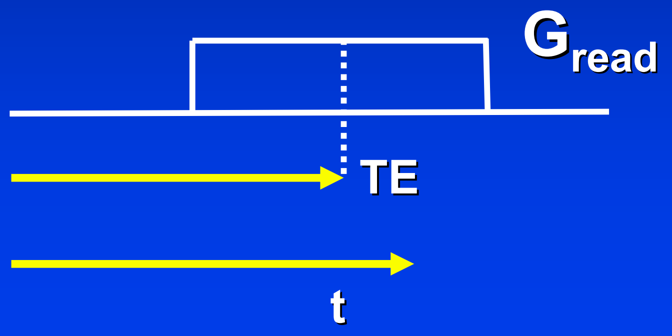
Gradient echo sequence



$$s(k)\bullet rect(k) \rightarrow s(k)\bullet rect(k)\bullet e^{-t/T_2^*}$$

Rewrite in terms of k:

$$k = \cancel{\gamma} G_r (t - TE)$$



Therefore:

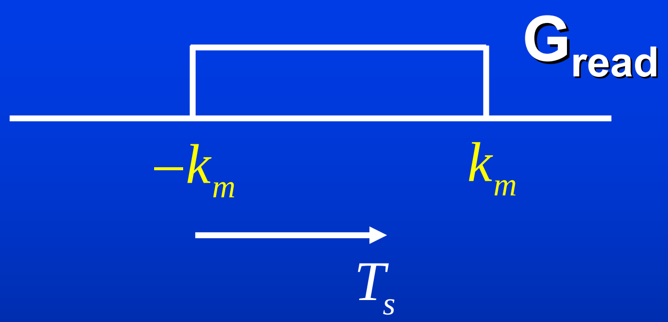
$$e^{-t/T_2^*} = e^{-TE/T_2^*} e^{-k/\cancel{\gamma} G T_2^*}$$

$$s(k) \bullet rect(k) \rightarrow s(k) \bullet rect(k) \bullet e^{-TE/T_2^*} e^{-k/\cancel{\gamma} G T_2^*}$$

$$PSF(x) = \mathfrak{I}^{-1}\left(rect(k) \bullet e^{-TE/T_2^*} e^{-k/\bar{\gamma} G T_2^*} \right)$$

$$= e^{-TE/T_2^*} \int_{-k_m}^{k_m} e^{2\pi i k x} e^{-k/\bar{\gamma} G T_2^*} dk$$

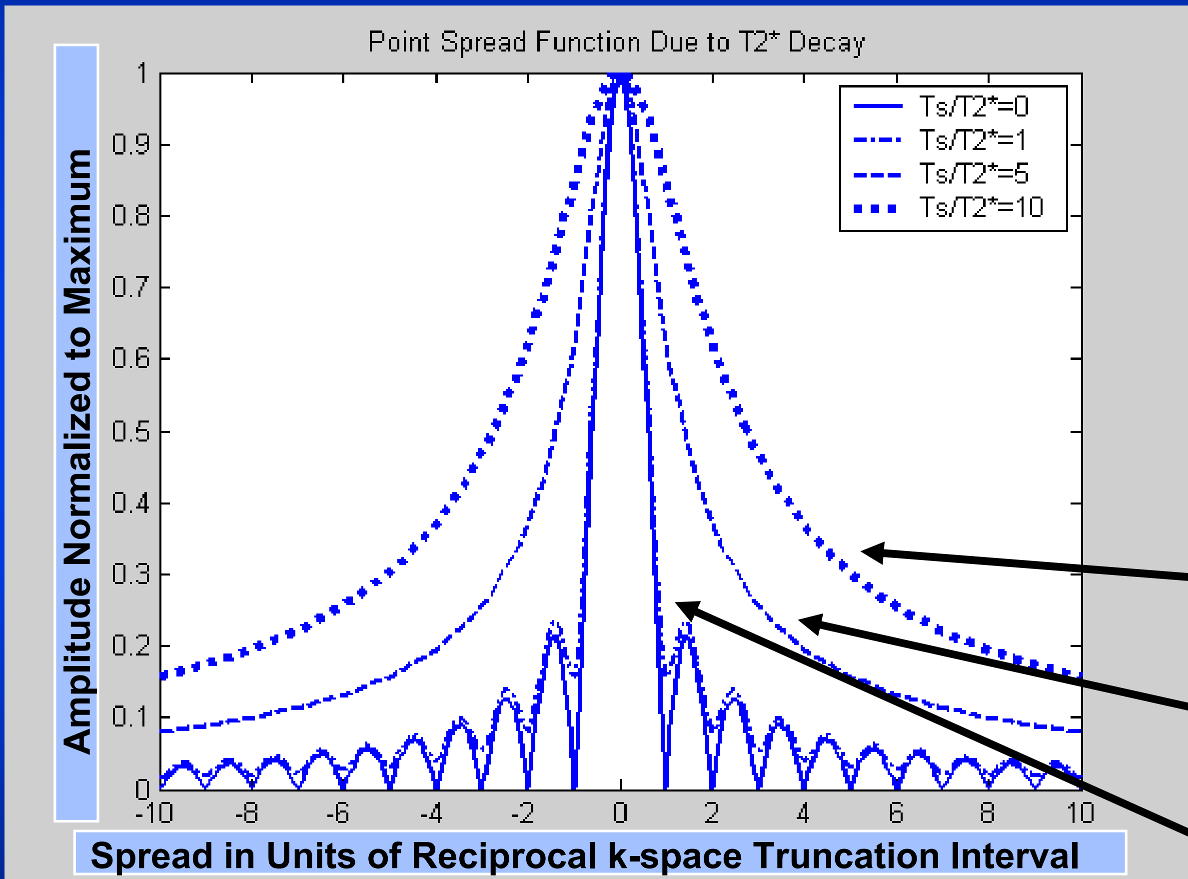
Note: $T_s = \dfrac{2k_m}{\bar{\gamma} G_{read}}$



Therefore:

$$PSF = PSF(x; T_s / T_2^*)$$

$$|PSF| = |PSF(x; T_s / T_2^*)|$$



Broadening occurs as data acquisition time lengthens on the time scale of T_2^* relaxation

$T_s = 10 \bullet T_2^*$; PSF dominated by relaxation

$T_s = 5 \bullet T_2^*$; Relaxation broadening

T_2^* negligible: PSF as for truncation

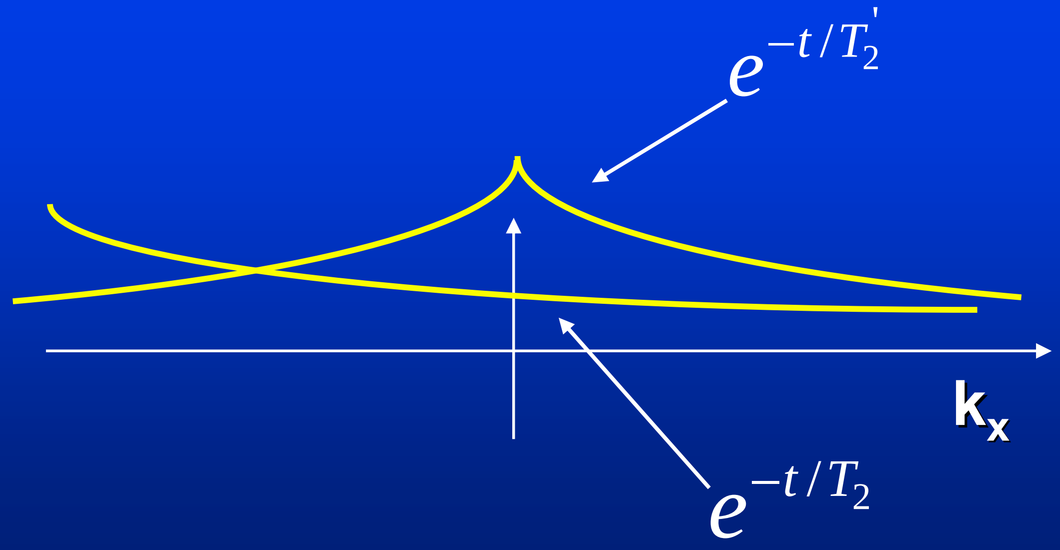
Point Spread Function Due to T_2 *decay*

$$1/T_2^* = 1/T_2 + 1/T_2'$$

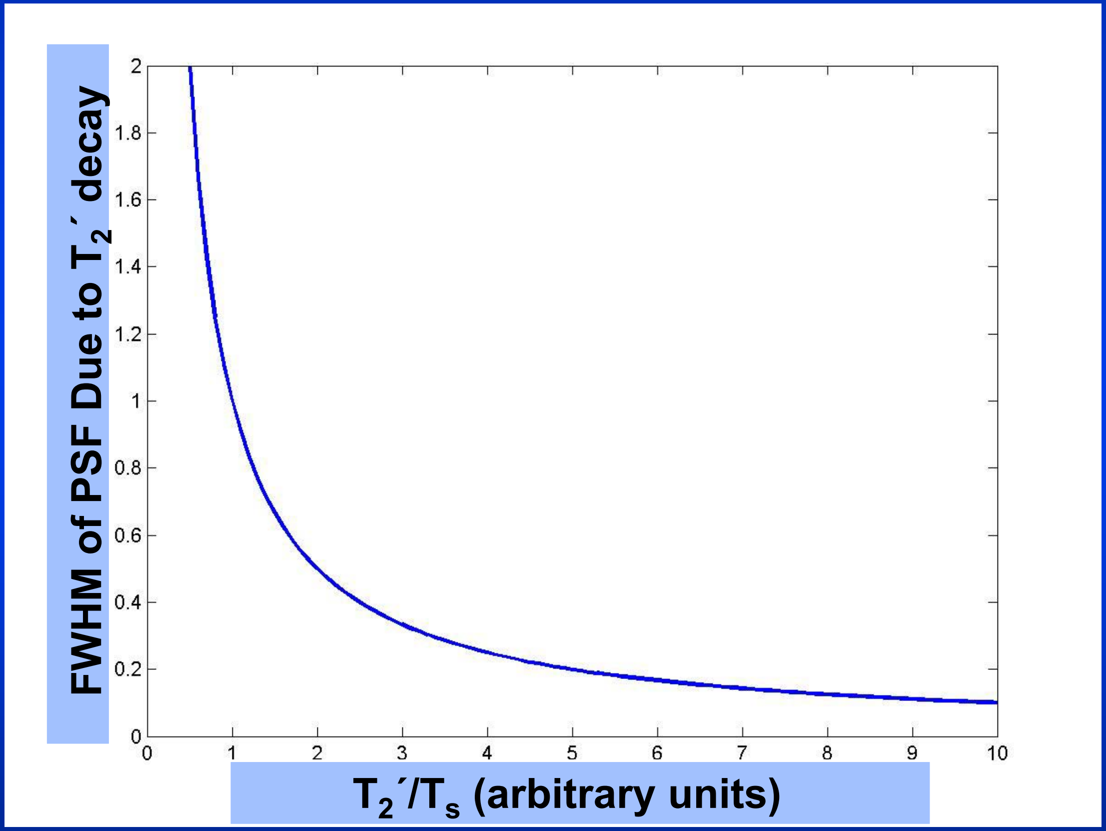
In the spin echo experiment

- T_2 decay starts from the beginning of each k-space line, at $-k_{max}$

- T_2' effectively "starts" at k = 0, in the middle of acquisition



This PSF can be described by its full width at half-maximum (FWHM)



Conclusions:

- The basic concepts of time/frequency signal processing can be carried over to x-space/k-space in MRI

- The imaging equation defines the relevant Fourier conjugate variables

- Δk_x and Δk_y are the sampling intervals, analogous to T_s

- Sampling and other operations on data are performed in k-space; the convolution theorem supplies the resulting effects on the image